Automorphisms of An Orthomodular Poset of Projections

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By using a lattice characterization of continuous projections defined on a topological vector space E arising from a dual pair, we determine the automorphism group of their orthomodular poset $Proj(E)$ by means of automorphisms and anti-automorphisms of the lattice L of all closed subspaces of E. A connection between the automorphism group of the ring of all continuous linear mappings defined on E and the automorphism group of the orthoposet $Proj(E)$ is established.

1. INTRODUCTION

In a vector space E, there exists a natural correspondence between the projections and pairs of subspaces: to every projection p is associated the pair $(Imp, Kerp)$ of subspaces. If E is a topological vector space and p a continuous projection then $(Imp, Kerp)$ is a pair of closed subspaces and $Imp + Kerp$ is a topological direct sum. In a previous paper Chevalier (2002), we introduced the projection poset $P(L)$ of a lattice L satisfying some properties of lattices of closed subspaces. We proved that if L is the lattice of all closed subspaces of a topological vector space E arising from a dual pair, then $P(L)$ is isomorphic to the poset of continuous projections defined on E (Theorem 1 and 2 of Chevalier (2002)). By using this isomorphism, we determined the automorphism group of a poset of continuous projections by means of automorphisms and anti-automorphisms of the lattice L (Theorem 3 of Chevalier (2002)).

This paper continues Chevalier (2002). In the first part, Theorem 3 is improved, a restrictive hypothesis is removed and its setting is extended to some incomplete lattices.

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In the second part of the paper, we prove a continuous form of the first fundamental theorem of projective geometry. This result allows us to relate the automorphism group of the orthomodular poset of continuous projections defined on a topological vector space E with the automorphism group of the ring of continuous linear mappings defined on E.

Information about the lattice concepts used in this paper may be found in Maeda and Maeda (1970), and Köthe (1969) or Schaefer (1966) are good references for topological vector spaces.

2. THE ORTHOMODULAR POSET OF PROJECTIONS OF A SYMMETRIC LATTICE

In this section, we recall some definitions and results from Chevalier (2002) where the reader is referred to for more information.

In a lattice L, $(a, b) \in L^2$ is a modular pair, written $(a, b)M$, if $(x \vee a) \wedge b = x \vee (a \wedge b)$ for every $x \leq b$. The pair (a, b) is a dual modular pair, written $(a, b)M^*$, if $(a, b)M$ holds in the dual lattice L^* of L and the lattice L is said to be a symmetric lattice if $(a, b)M$ implies $(b, a)M$ and $(a, b)M^*$ implies $(b, a)M^*$.

Our purpose in the following definition of the projection poset of a lattice L is to obtain, when L is the lattice of all closed subspaces of a topological vector space E, a poset defined in a algebraic setting and isomorphic to the poset of all continuous linear projections defined on E. See Chevalier (2002) for a discussion about the motivation of this definition.

Definition 1. Let L be a symmetric lattice. The projection poset $P(L)$ of L is the following subset of the direct product $L \times L$:

$$P(L) = \{(a, b) \in L \times L \mid a \vee b = 1, \ a \wedge b = 0, \ (a, b)M, \ (a, b)M^*\}$$

For a projection $p = (a, b) \in P(L)$, a is called the image of p and b its kernel. If (a, b) is a projection of a symmetric lattice L then (b, a) is also a projection and we write $(b, a) = (a, b)^{\perp}$.

Proposition 1. *(Chevalier (2002), Harding (1996), Mushtari (1998)) Let L be a symmetric lattice with 0 and 1. If $P(L)$ is ordered by the restriction $\leq$ of the order relation on $L \times L^*$ then $(P(L), \leq, \perp)$ is an orthomodular poset (abbreviated OMP). If L possesses a structure of OMP then this OMP is naturally isomorphic to a suborthomodular poset of $P(L)$.*

An AC-lattice is an atomistic lattice with the covering property : if p is an atom and $a \wedge p = 0$ then $\prec a \vee p$, that is $a \leq x \leq a \vee p$ implies $a = x$ or $a \vee p = x$.

If L and L^* are AC-lattices, L is called a DAC-lattice. Any DAC-lattice is symmetric and finite-modular (Maeda and Maeda, 1970, Theorem 27.5). Irreducible complete DAC-lattices of length ≥ 4 are representable by lattices of closed subspaces and many lattices of subspaces are DAC-lattices. We will now specify this last assertion.

Let K be a field, E a left vector spaces over K, F a right vector space over K. If there exists a nondegenerate bilinear form $\mathcal{B}$ on $E \times F$, we say that (E, F) is a pair of dual spaces. For example, if E is a locally convex space and E' its topological dual space then (E, E') is naturally a pair of dual spaces with $\mathcal{B}(x, y) = y(x)$ (Köthe, 1969, p. 234).

For a subspace A of E, we put

$$A^\perp = \{y \in F \mid \mathcal{B}(x, y) = 0 \text{ for every } x \in A\}.$$

Similarly, let

$$B^\perp = \{x \in E \mid \mathcal{B}(x, y) = 0 \text{ for every } y \in B\}$$

for every subspace B of F. A subspace A of E is called F-closed if $A = A^{\perp\perp}$ and the set of all F-closed subspaces, denoted by $L_F(E)$ and ordered by set-inclusion, is a complete irreducible DAC-lattice. Conversely, for any irreducible complete DAC-lattice L of length ≥ 4, there exists a pair (E, F) of dual spaces such that L is isomorphic to the lattice of all F-closed subspaces of E (Maeda and Maeda, 1970, Theorem 33.7).

The set of all E-closed subspaces of F is similarly defined and is also a DAC-lattice.

Let (E, F) be a pair of dual spaces. The linear weak topology on E, denoted by $\sigma(E, F)$, is the linear topology defined by taking $\{G^\perp \mid G \subset F, \ \dim G < \infty\}$ as a basis of neighbourhoods of 0. If F is interpreted as a subspace of the algebraic dual of E then a sub-basis of neighbourhoods of 0 consists of kernels of elements of F.

The linear weak topology on F, noted $\sigma(F, E)$, is defined in the same way. The space F can be interpreted as the topological dual of E for the $\sigma(E, F)$ topology and E as the topological dual of F for the $\sigma(F, E)$ topology. Equipped with their linear weak topologies, E and F are topological vector spaces (Köthe, 1969, § 10.3) if the topology on K is discrete.

Moreover, for a subspace $G \subset E$, we have $\bar{G} = G^{\perp\perp}$ and thus a closed subspace in E is an unambiguous notion.

The following theorem shows that our definition of a projection poset of a lattice is appropriate for our purpose.

Theorem 1. *(Chevalier, 2002) Let L be a complete irreducible DAC-lattice. If L is representable as the lattice $\mathcal{L}$ of all F-closed subspaces of a pair of dual spaces*

(E, F) then the projection orthoposets $P(L)$ and $P(\mathcal{L})$ are isomorphic and the correspondence $p \mapsto (Imp, Kerp)$ is an isomorphism between the orthomodular poset of $\sigma(E, F)$-continuous linear projections defined on E onto $P(\mathcal{L})$

The linear weak topology seems to be a poor topology. However, a linear mapping f, defined on a locally convex space E, is weakly continuous if and only f is continuous for the linear weak topology $\sigma(E, E')$ (Köthe, 1969, 20.4) and so we obtain the following consequences of Theorem 1.

Corollary 1. *(Chevalier, 2002)* *Let E be a locally convex space and L its lattice of all closed subspaces. The projection orthomodular poset $P(L)$ is isomorphic to the poset of weakly continuous linear projections defined on E.*

Corollary 2. *(Chevalier, 2002)* *If H is a Hilbert space (more generally, a metrizable space) and L its lattice of closed subspaces then the projection orthomodular poset $P(L)$ is isomorphic to the orthoposet of continuous linear projections defined on H.*

3. AUTOMORPHISMS OF AN ORTHOMODULAR POSET OF PROJECTIONS

The main result of Chevalier (2002) is the generalization of a theorem of Ovchinnikov (1993) and gives a description of automorphisms of a projection orthoposet $P(L)$ by means of automorphisms and anti-automorphisms of the lattice L when L is a complete DAC-lattice satisfying the condition

for every $a \in L$ there exists $b \in L$ such that $(a, b) \in P(L)$ (C).

Moreover, it is proved in Chevalier (2002) that there are exactly two kinds of automorphisms on an orthoposet of projections: the so-called even automorphisms which transform projections with the same image into projections with the same image and the odd automorphisms which transform projections with the same image into projections with the same kernel. This fact generalizes a theorem of Ovchinnikov (1993).

In this section, we will improve on the main result of Chevalier (2002) by removing the restriction condition (C) and by extending its setting to certain incomplete lattices.

Let us say that an irreducible DAC-lattice L is a G-lattice if L is complete or if L is modular and complemented. Typical examples of G-lattices are obtained by considering a Hilbert space H: the lattice of all closed subspaces of H is a G-lattice as a complete irreducible DAC-lattice and its sublattice of finite or cofinite dimensional elements is a G-lattice as a complemented modular irreducible

DAC-lattice. Irreducible DAC-lattices of length ≥ 4 which are either complete or modular and complemented share the following properties:

- Every atom has more than one complement.
- If $a \prec b$ then there exist different atoms p_1 and p_2 such that $a \vee p_1 = a \vee p_2 = b$.
- Two different atoms have a common complement.

By using these facts, all the results preceding Theorem 3 of Chevalier (2002), proved for irreducible complete DAC-lattices, extend to G-lattices and an improved version of Theorem 3 is as follows.

Theorem 2.   *Let L be a G-lattice of length ≥ 4. For every automorphism ϕ of the poset $P(L)$ there exists*

1. *an automorphism f of the lattice L such that $\phi((a, b)) = (f(a), f(b))$, $(a, b) \in P(L)$, if ϕ is even,*
2. *an anti-automorphism g of the lattice L such that $\phi((a, b)) = (g(b), g(a))$, $(a, b) \in P(L)$, if ϕ is odd.*

Conversely, if f is an automorphism of L then $\phi : P(L) \mapsto L \times L^*$ defined by $\phi((a, b)) = (f(a), f(b))$ is an even automorphism of $P(L)$ and if g is an anti-automorphism of L then $\psi : P(L) \mapsto L \times L^*$ defined by $\psi((a, b)) = (g(b), g(a))$ is an odd automorphism of $P(L)$.

Proof: First we recall some notations from Chevalier (2002). In a DAC-lattice L, $At(L)$ denotes the set of all atoms, $At^*(L)$ is the set of all coatoms, and $\mathcal{F}(L)$ is the G-lattice of all finite or cofinite elements of L. By $P_1(L)$ we mean the set of all atoms of the projection poset $P(L)$.

Let us denote by L^+ the lattice L if L is an irreducible complemented modular DAC-lattice and the lattice $\mathcal{F}(L)$ if L is a complete irreducible DAC-lattice. In the two cases, L^+ is an irreducible complemented modular DAC-lattice and the restriction of ϕ to $P(L^+)$ is an automorphism.

Assume that ϕ is even. By Proposition 8 of Chevalier (2002), there exist two bijections $f_1 : At(L^+) \mapsto At(L^+)$ and $f_2 : At^*(L^+) \mapsto At^*(L^+)$ such that, for every $(p, q) \in P_1(L^+)$, $\phi((p, q)) = (f_1(p), f_2(q))$. Let $a \in L^+$, $a \neq 0$. There exists $b \in L^+$ such that $(a, b) \in P(L^+)$. For any atom $p \leq a$ there exists a coatom q with $(p, q) \leq (a, b)$. Thus $\phi((p, q)) = (f_1(p), f_2(q)) \leq \phi(a, b)$. If $\phi((a, b)) = (c, d)$ then

$$f_1(\{p \in At(L^+) \mid p \leq a\}) \subset \{p \in At(L^+) \mid p \leq c\}.$$

By using ϕ^{-1}, we have

$$f_1(\{p \in At(L^+) \mid p \leq a\}) = \{p \in At(L^+) \mid p \leq c\}$$

and so Proposition 9 of Chevalier (2002) implies that f_1 can be extended to an automorphism $\bar{f}_1$ of the lattice L^+. Similarly, f_2 has an extension, $\bar{f}_2$.

The correspondence $(a, b) \in P(L^+) \mapsto (\bar{f}_1(a), \bar{f}_2(b))$ is an automorphism of the poset $P(L^+)$ which agrees with ϕ on $P_1(L^+)$. As $P(L^+)$ is atomistic (Lemma 6 of Chevalier (2002)) , for every $(a, b) \in P(L^+)$, we have $\phi((a, b)) = (\bar{f}_1(a), \bar{f}_2(b))$. This equality is also true for $(a, b) = (0, 1)$.

By Proposition 6 of Chevalier (2002), ϕ is also an automorphism of the orthoposet $P(L^+)$ and thus $\phi((a, b)^\perp) = \phi((a, b))^\perp$, that is $(\bar{f}_1(b), \bar{f}_2(a)) = (\bar{f}_2(b), \bar{f}_1(a))$ and so $\bar{f}_1 = \bar{f}_2$.

The proof is similar if ϕ is odd and is complete if L is a complemented modular DAC-lattice. If L is an irreducible complete DAC-lattice, a lemma is necessary. $\qquad\square$

Lemma 1. *Let L be an irreducible complete DAC-lattice of length ≥ 4. Any automorphism of the lattice $\mathcal{F}(L)$ extends to an automorphism of L.*

Proof: Let (E, F) be a pair of dual spaces such that L is isomorphic to the lattice $L_F(E)$ of all F-closed subspaces of E. The lemma will be proved if any automorphism ψ of $\mathcal{F}(L_F(E))$ extends to an automorphism of $L_F(E)$.

Define, for every subspace N of E, $\varphi(N) = \bigcup \{\psi(M) \mid M \subset N, \dim M < \infty\}$. It is clear that $\varphi(N)$ is a subspace of E. Let X be a subspace of E and $N = \bigcup \{\psi^{-1}(M) \mid M \subset X, \dim M < \infty\}$. The set N is a subspace of E and we have $\varphi(N) = \bigcup \{\psi(\psi^{-1}(M)) \mid M \subset X, \dim M < \infty\} = X$.

Let M, N be two subspaces of E. If $M \subset N$ then $\varphi(M) \subset \varphi(N)$ and, for the converse, let L be a subspace of M with $\dim L = 1$. We have $\psi(L) \subset \varphi(M) \subset \varphi(N)$ and if $0 \neq x \in \psi(L)$ then there exists a subspace $K \subset N$, $\dim K < \infty$, such that $x \in \psi(K)$. By $\dim \psi(L) = 1$, we have $\psi(L) \subset \psi(K)$ and therefore $L \subset K \subset N$. Finally, $M \subset N$ and φ is an automorphism of the lattice of all subspaces of E. This automorphism extends ψ since, for $M \in L_F(E)$, $\psi(M)$ and $\varphi(M)$ have the same finite dimensional subspaces.

Let $M \in L_F(E)$. As $L_F(E)$ is a DAC-lattice there exists a family (H_α) of F-closed hyperplanes such that $M = \bigwedge H_\alpha = \bigcap H_\alpha$ and

$$\varphi(M) = \varphi\Big(\bigcap H_\alpha\Big) = \bigcap \varphi(H_\alpha) = \bigcap \psi(H_\alpha) = \bigwedge \psi(H_\alpha).$$

Therefore $\varphi(M)$ is F-closed and, as $\varphi^{-1}(M)$ is also F-closed, φ is an automorphism of $L_F(E)$ extending ψ.

We return to the proof of the theorem. If ϕ is an even automorphism of $P(L)$, L an irreducible complete DAC-lattice, then ϕ is also an automorphism of $P(\mathcal{F}(L))$ and so there exists an automorphism f of $\mathcal{F}(L)$ such that $\phi(a, b) = (f(a), f(b))$ for any $(a, b) \in \mathcal{F}(L)$. By using the lemma, f extends to an automorphism of

the lattice L and, as $P(L)$ is an atomistic lattice, $\phi(a, b) = (f(a), f(b))$ for any $(a, b) \in P(L)$.

The proof is similar if ϕ is odd.

For the converse, Proposition 5 of Chevalier (2002) shows that it suffices to prove that $(a, b) \in P(L)$ implies $(f(a), f(b)) \in P(L)$ for any automorphism f and $(g(b), g(a)) \in P(L)$ for any anti-automorphism g. But these implications are an easy consequence of the equivalence:

$$(a, b)M \Leftrightarrow \forall x \in L, \ ((x \wedge b) \vee a) \wedge b = (x \wedge b) \vee (a \wedge b)$$

and

$$(a, b)M^* \Leftrightarrow \forall x \in L, \ ((x \vee b) \wedge a) \vee b = (x \vee b) \wedge (a \vee b). \qquad \square$$

4. MORE ABOUT AUTOMORPHISMS

By Theorem 2, the automorphisms of the projection poset $P(L)$ of a complete irreducible DAC-lattice L of length ≥ 4 are determined by the automorphisms and the anti-automorphisms of the lattice L. As every complete irreducible DAC-lattice of length ≥ 4 is the lattice of all closed subspaces of a pair of dual spaces, in this section we will investigate the automorphism group of the lattice of closed subspaces.

4.1. A Continuous Form of the First Fundamental Theorem of Projective Geometry

If E_1 and E_2 are vector spaces of dimensions at least 3 over the fields K_1 and K_2 then, by the first fundamental theorem of projective geometry (Baer, 1952, p. 44, Varadaradjan, 1985, p. 21), the lattices of all subspaces of E_1 and E_2 are isomorphic if and only if K_1 and K_2 are isomorphic fields and E_1 and E_2 have the same dimension. Moreover, if ψ is an isomorphism from the lattice of all subspaces of E_1 onto the lattice of all subspaces of E_2 then there exists a semi-linear bijection $s : E_1 \mapsto E_2$ such that, for every subspace $M \subset E_1$, $\psi(M) = s(M)$. Conversely, every semi-linear bijection of E_1 onto E_2 induces a lattice isomorphism.

In the following proposition, we generalize a part of the previous result to lattices of closed subspaces.

Proposition 2. *Let (E_1, F_1) and (E_2, F_2) be two pairs of dual spaces over the fields K_1 and K_2. If there exists an isomorphism ψ of the lattice $L_{F_1}(E_1)$ onto the lattice $L_{F_2}(E_2)$ then K_1 and K_2 are isomorphic fields and there exists a semi-linear bijection $s : E_1 \mapsto E_2$ such that, for every F_1-closed subspace M of E_1, $\psi(M) = s(M)$.*

Proof: The mapping ψ is an order isomorphism of the poset of all finite dimensional subspaces of E_1 onto the poset of all finite dimensional subspaces of E_2.

Define, for every subspace N of E_1, $\varphi(N) = \bigcup\{\psi(M) \mid M \subset N, \dim M < \infty\}$. By a proof similar to the proof of Lemma 1, φ is an isomorphism of the lattice of all subspaces of E_1 onto the lattice of all subspaces of E_2 which extends ψ. Thus, by the first fundamental theorem of projective geometry, the fields K_1 and K_2 are isomorphic and there exists a semi-linear bijection $s : E_1 \mapsto E_2$ such that, for every F_1-closed subspace M of E_1, $\psi(M) = s(M)$. $\square$

Remark 1. This proof is similar to the proof of Lemma 1 of Fillmore and Longstaff (1984) where the authors prove the same result for complex normed spaces.

In the case of lattices of all subspaces of vector spaces, any semi-linear bijection induces a lattice isomorphism. For lattices of closed subspaces, only continuous semi-linear bijections are allowed.

Proposition 3. *Let (E_1, F_1) and (E_2, F_2) be two pairs of dual spaces over the same field. If E_1 and E_2 are equipped, respectively, with the $\sigma(E_1, F_1)$-topology and the $\sigma(E_2, F_2)$-topology then, for every semi-linear bijection $s : E_1 \mapsto E_2$, the following statements are equivalent.*

1. *The bijection s is bicontinuous (i.e. both s and s^{-1} are continuous).*
2. *$H \in L_{F_1}(E_1) \mapsto s(H)$ is a bijection from the set of all F_1-closed hyperplanes of E_1 onto the set of all F_2-closed hyperplanes of E_2.*
3. *$M \in L_{F_1}(E_1) \mapsto s(M)$ is an isomorphism from the lattice $L_{F_1}(E_1)$ onto $L_{F_2}(E_2)$.*

Proof: 1) $\Rightarrow$ 2). Since s is a semi-linear bijection, the correspondence $M \mapsto s(M)$ is an isomorphism of the lattice of all subspaces of E_1 onto the lattice of all subspaces of E_2 and maps bijectively the sets of all hyperplanes. If $H \subset E_2$ is an F_2-closed hyperplane then H is a neighbourhood of 0 for the $\sigma(E_2, F_2)$ topology. Since s is continuous, there exists a finite dimensional subspace $G \subset F_1$ such that $G^\perp \subset s^{-1}(H)$. As $s^{-1}(H)$ has a finite codimension in $G^\perp$, $s^{-1}(H)$ is closed (Köthe, 1969, property (7), p. 87) and since s^{-1} is also continuous, $H \mapsto s(H)$ is a bijection from the set of all F_1-closed hyperplanes of E_1 onto the set of all F_2-closed hyperplanes of E_2.

2) $\Rightarrow$ 3). Let $M \in L_{F_1}(E_1)$. As $L_{F_1}(E_1)$ is a DAC-lattice there exists a family (H_α) of F_1-closed hyperplanes such that $M = \bigwedge H_\alpha = \bigcap H_\alpha$ and therefore $s(M) = s(\bigcap H_\alpha) = \bigcap s(H_\alpha) = \bigwedge s(H_\alpha)$. Thus $s(M) \in L_{F_2}(E_2)$ and $s(L_{F_1}(E_1)) \subset L_{F_2}(E_2)$. As s^{-1} also satisfies the statement (2), $s(L_{F_1}(E_1)) =$

$L_{F_2}(E_2)$ and $M \in L_{F_1}(E_1) \mapsto s(M)$ is an isomorphism from the lattice $L_{F_1}(E_1)$ onto $L_{F_2}(E_2)$

3) $\Rightarrow$ 1). This is clear since the family of all closed hyperplanes is a 0-neighbourhood sub-basis for the linear weak topology. $\square$

Corollary 3. *Let E_1 and E_2 be real metrizable locally convex spaces.*

1. *E_1 and E_2 are isomorphic if and only if their lattices of closed subspaces $\mathcal{C}(E_1)$ and $\mathcal{C}(E_2)$ are isomorphic.*
2. *$\psi : \mathcal{C}(E_1) \mapsto \mathcal{C}(E_2)$ is a lattice isomorphism if and only if there exists a bicontinuous linear bijection $s : E_1 \mapsto E_2$ such that, for every $M \in \mathcal{C}(E_1)$, $\psi(M) = s(M)$.*

Proof: As E_1 and E_2 are real vector spaces, semi-linear bijections are simply linear bijections and, as E_1 and E_2 are metrizable locally convex spaces, a linear mapping $s : E_1 \mapsto E_2$ is continuous if and only if s is a continuous mapping for the linear weak topologies. $\square$

Remark 2. This corollary is a generalization of the following result of Mackey (1942): two real normed spaces X_1 and X_2 are isomorphic if and only if there exists a linear bijection $T : X_1 \mapsto X_2$ which carries bijectively closed hyperplanes of X_1 into closed hyperplanes of X_2; if T exists then T is bicontinuous. This result is extended to complex normed spaces in Fillmore and Longstaff (1984): if $\psi : \mathcal{C}(X) \mapsto \mathcal{C}(Y)$ is an isomorphism of the lattices of closed subspaces of infinite dimensional complex normed spaces X and Y then there exists a bicontinuous linear or conjugate linear bijection $s : X \mapsto Y$ such that $\psi(M) = s(M)$ for all $M \in \mathcal{C}(X)$. By using the following theorem, this last result is also a consequence of Proposition 3: if $s : X \mapsto Y$ is a bijective semi-linear transformation of infinite-dimensional complex normed spaces that carries closed hyperplanes to closed hyperplanes then s is either linear or conjugate linear (Kakutani and Mackey, 1946, or Fillmore and Longstaff (1984), Lemma 2).

4.2. Automorphism Group of a Projection Poset

Let (E, F) be a pair of dual spaces. If L is the DAC-lattice of all F-closed subspaces of E then Theorem 2 and Proposition 3 allow one to obtain a description of even automorphisms of the projection lattice $P(L)$ by means of lattice automorphisms of L. They are all the correspondences $(M, N) \in P(L) \mapsto (s(M), s(N))$ where $s : E \mapsto E$ fulfills the equivalent conditions of Proposition 3. In the language of rings, even automorphisms of the OMP of all continuous linear projections are of the form $p \mapsto sps^{-1}$ since, for a linear projection p, $s(Im\ p) = Im\ sps^{-1}$ and $s(Ker\ p) = Ker\ sps^{-1}$.

If g is an anti-automorphism of L then any anti-automorphism is of the form fg where f is some automorphism. Thus the set of all anti-automorphisms is determined by a particular anti-automorphism and the group of all automorphisms but it seems difficult to find conditions assuring the existence of an anti-automorphism of $L = L_F(E)$. We will now discuss this point.

A first case is well known : if L is the modular lattice of all subspaces of an infinite-dimensional vector space E then L has no anti-automorphism (Baer, 1952, Self-duality theorem, p. 97). This results is extended in Ornstein (1959) to infinite dimensional projective geometries that are irreducible, complemented, modular, complete atomic lattices of infinite length. Such lattices can be represented as lattices of closed subspaces of pairs of dual spaces of infinite dimension in which any sum of two closed subspaces is closed (Ornstein, 1959). Thus the automorphism group of the OMP of projections defined on an infinite-dimensional projective geometry L is isomorphic to the automorphism group of the lattice L.

Now suppose that E is finite dimensional. As F is isomorphic to a subspace of E^* and E to a subspace of F^*, we can assume that $E = F$ with dim $E = n$. The existence of an anti-automorphism of L is equivalent to the existence of an anti-automorphism of the field K: if α is an anti-automorphism of K and if $(e_i)_{1 \leq i \leq n}$ is a basis of E then the α-bilinear form

$$\left(\sum x_i e_i, \sum y_i e_i \right) \mapsto \left\langle \sum x_i e_i, \sum y_i e_i \right\rangle = \sum_{i=1}^{n} x_i \alpha(y_i)$$

is non-degenerate and determines an anti-automorphism g of L.

The anti-automorphism α is involutary if and only if g is involutary and, in this case, the group formed by the automorphisms and the anti-automorphisms of L is the semi-direct product of the normal subgroup of all automorphisms and the subgroup $\{1_E, g\}$. The automorphism group of $P(L)$ is the semi-direct product of the normal subgroup of even automorphisms and a two-element subgroup $\{1_{P(L)}, \gamma\}$ where γ is an involutary odd automorphism.

In the infinite dimensional case and if $E = F$ then a particular anti-automorphism is given by

$$X \in L_E(E) \mapsto X^{\perp} = \{x \in E \mid \langle x, X \rangle = 0\}$$

and the previous results allow one to obtain all the automorphisms and all the anti-automorphisms of $L_E(E)$ and thus to determine the automorphism group of its projection lattice.

Example 1. If H is a Hilbert space then the two pairs of dual spaces (H, H) and (H, H') coincide. The correspondence $X \in L_H(H) \mapsto X^{\perp}$ is an involutary anti-automorphism of the lattice of all closed subspaces of H and $p \mapsto p^*$ is the corresponding involutary odd automorphism of the orthoposet of continuous

linear projection defined on H. By using the previous results, we find again the main result of Ovchinnikov (1993): the automorphisms of the orthoposet *proj*(H) of all continuous linear projections of H are of the form $p \mapsto s^{-1}ps$ or $p \mapsto s^{-1}p^*s$ where s is a continuous linear bijection in the real case and a continuous linear or conjugate linear bijection in the infinite dimensional complex case. In the finite dimensional complex case, s is only a semi-linear bijection. The automorphism group of *Proj*(H) is a semi-direct product of the normal subgroup of even automorphisms and a two-element subgroup.

4.3. Application to the Ring of Continuous Linear Mappings

The following proposition is proved in Jacobson (1964) (Isomorphism Theorem, p. 79) in the study of primitive rings having minimal right ideals. In Fillmore and Longstaff (1984), a different proof is given in the particular case of infinite-dimensional complex normed linear spaces. Here, we generalize the latter proof for two pairs of dual spaces and obtain the result of Jacobson (1964).

Proposition 4. *Let (E_1, F_1) and (E_2, F_2) be the two pairs of dual spaces (over the fields K_1 and K_2) and let us denote by $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ the rings of all continuous linear mappings defined on E_1 and E_2 equipped with their linear weak topologies. If there exists an isomorphism of rings, $\Phi : \mathcal{B}(E_1) \mapsto \mathcal{B}(E_2)$, then K_1 and K_2 are isomorphic fields and there exists a bicontinuous semi-linear bijection $S : E_1 \mapsto E_2$ such that, for every $T \in \mathcal{B}(E_1)$,*

$$\Phi(T) = STS^{-1}.$$

Proof: If p is a continuous projection then, as projections are defined by means of an equation in the language of rings, $\Phi(p)$ is also a continuous projection. The same argument shows that, for two projections p and q, we have $p \leq q$ if and only if $\Phi(p) \leq \Phi(q)$. Moreover $\Phi(1_{E_1} - p) = 1_{E_2} - \Phi(p)$ and the restriction of Φ to the set of all continuous projections is an orthoposet isomorphism.

Fix a linear projection $p \in \mathcal{B}(E_1)$ of rank 1 (such projection exists since the projection lattice of the DAC-lattice $L_{F_1}(E_1)$ is atomistic with atoms of the form (X, Y), X a one dimensional subspace) and non-zero elements $x_0 \in Imp$, $y_0 \in Im\ \Phi(p)$. Remark that $\Phi(p)$ is also a continuous projection of rank 1.

Let $x \in E_1$ and consider the linear mapping U defined by $U(x_0) = x$ and $U(t) = 0$ for $t \in Ker\ p$. The mapping U is continuous as a linear mapping with a finite-dimensional range and a closed kernel (Schaefer, 1966, p. 75).

Assume that $V \in \mathcal{B}(E_1)$ also satisfies $V(x_0) = x$. For every $\lambda \in K_1$, $U(\lambda x_0) = V(\lambda x_0)$ and thus $U \circ p = V \circ p$. We have $\Phi(U) \circ \Phi(p) = \Phi(V) \circ \Phi(p)$ and so $\Phi(U)(y_0) = \Phi(V)(y_0)$. Thus, we can define a mapping $S : E_1 \mapsto E_2$ by $S(x) = \Phi(U)(y_0)$.

Let x, $x' \in E_1$ and U, U', W be elements of $\mathcal{B}(E_1)$ such that $U(x_0) = x$, $U'(x_0) = x'$, $W(x_0) = x + x'$. As $(U + U')(x_0) = x + y$, we have $(U + U') \circ p = W \circ p$ and $S(x) + S(x') = S(x + x')$.

In a similar way, it can be proved that S is a bijection and $\Phi(T) = STS^{-1}$, for every $T \in \mathcal{B}(E_1)$.

The center of the rings $\mathcal{B}(E_i)$, $i = 1$, 2 is $\{k \, 1_{E_i} \mid k \in K_i\}$ since a linear mapping which commutes with all projections of rank 1 is a homothetic transformation. Therefore, for every $k \in K_1$, there exists $k' \in K_2$ such that $\Phi(k 1_{E_1}) = k' 1_{E_2}$. One can check that the mapping $\sigma : K_1 \mapsto K_2$ defined by $\sigma(k) = k'$ is an isomorphism from the field K_1 onto the field K_2 and that $S(\lambda x) = \sigma(\lambda)S(x)$ for every $\lambda \in K_1$.

The last step is the proof of continuity of S. Let $f \in F_1$ be a continuous non-zero linear form on E_1. Define a linear mapping $T : E_1 \mapsto E_1$ by $T(x) = f(x)x_0$. The mapping $x \in E_1 \mapsto (f(x), x_0) \in K \times E_1$ is continuous and, as E_1 is a topological vector space, $T \in \mathcal{B}(E_1)$. We have $STS^{-1} = \Phi(T) \in \mathcal{B}(E_2)$ and

$$x \in \mathit{Ker} \; STS^{-1} \Leftrightarrow TS(x) = 0 \Leftrightarrow f(S(x)) = 0$$

$$\Leftrightarrow S(x) \in \mathit{Ker} \; f \Leftrightarrow x \in S(\mathit{Ker} \; f)$$

Since $STS^{-1} = \Phi(T)$ is continuous, $\mathit{Ker} \; STS^{-1}$ is closed and S, which carries hyperplanes to hyperplanes, carries closed hyperplanes to closed hyperplanes. The mapping S^{-1} is continuous and, by symmetry, so does S. □

Remark 3. Assume that E_1 and E_2 are real locally convex spaces and that F_1 and F_2 are their topological duals for the weak topology. The rings $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ are the rings of weakly continuous linear mappings defined on E_1 and E_2. Every ring isomorphism $\Phi : \mathcal{B}(E_1) \mapsto \mathcal{B}(E_2)$ is of the form $\Phi(T) = STS^{-1}$ with $S : E_1 \mapsto E_2$ a weakly bicontinuous linear bijection. If E_1 and E_2 are metrizable then S is continuous (Schaefer, 1966, Chap.IV, 7.4). For real Banach space this result is due to Eidelheit (1940).

Now assume that E_1 and E_2 are infinite dimensional complex normed spaces and $F_1 = E_1'$, $F_2 = E_2'$. As S carries closed hyperplanes to closed hyperplanes, S is linear or conjugate linear (Kakutani and Mackey, 1946, Lemma 2) and by Fillmore and Longstaff (1984), Lemma 3, S is bicontinuous. We have obtained a result of Arnold (1944) (see also Fillmore and Longstaff, 1984, Theorem 2): if Φ is an isomorphism of the rings of continuous linear transformations on infinite dimensional complex normed spaces E_1 and E_2 then there exists a bicontinuous linear or conjugate linear bijection $S : E_1 \mapsto E_2$ such that $\Phi(T) = STS^{-1}$.

Proposition 5. *Let (E, F) be a pair of dual spaces.*

1. *The restriction of an automorphism of the ring $\mathcal{B}(E)$ to the set of continuous linear projection is an even orthoposet automorphism and the*

restriction of an anti-automorphism of the ring $\mathcal{B}(E)$ to the set of contin-
uous linear projection is an odd orthoposet automorphism.

2. *Conversely, every even automorphism of the orthoposet of continuous linear projections defined on E extends to an automorphism of the ring $\mathcal{B}(E)$.*

Proof: 1) If ϕ is an automorphism or an anti-automorphism of the ring $\mathcal{B}(E)$ then its restriction to the set of continuous linear projections is an orthoposet automorphism. The nature of this automorphism will be given by the following lemma.   □

Lemma 2.   *Let p and q be the two linear projections defined on a vector space E.*

1. *$Im\ p = Im\ q \Leftrightarrow pq = q$ and $qp = p$.*
2. *$Ker\ p = Ker\ q \Leftrightarrow pq = p$ and $qp = q$.*

Proof: If $pq = q$ and $x \in Im\ q$ then $q(x) = x$ and, since $p(q(x)) = q(x)$, we have $p(x) = x$ and $x \in Im\ p$. Conversely, if $Im\ q \subset Im\ p$ and $x \in E$ then $x = x_1 + x_2$ with $x_1 \in Im\ q$ and $x_2 \in Ker\ q$. We have $p(q(x)) = p(q(x_1)) = p(x_1) = x_1 = q(x)$ and thus $p(q(x)) = q(x)$. Finally, $Im\ q \subset Im\ p \Leftrightarrow pq = q$ and all the other proofs are similar.

We return to the proof of the proposition. If ϕ is an automorphism of $\mathcal{B}(E)$ then, for two projections p and q,

$$Im\ p = Im\ q \Leftrightarrow pq = q \ \text{ et } \ qp = p$$
$$\Leftrightarrow \phi(p)\phi(q) = \phi(q) \ \text{ et } \ \phi(q)\phi(p) = \phi(p)$$
$$\Leftrightarrow Im\ \phi(p) = Im\ \phi(q),$$

and the restriction of ϕ to the set of continuous linear projections is an even orthoposet automorphism. By a similar proof, the restriction to the set of continuous linear projections of an anti-automorphism is an odd orthoposet automorphism.

If Ψ is an even orthoposet automorphism of the set of all continuous linear projection defined on E then there exists an automorphism f of the lattice of all closed subspaces of E such that $\Psi(Im\ p, Ker\ p) = (f(Im\ p), f(Ker\ p))$. Let S be the bicontinuous semi-linear bijection such that $S(X) = f(X)$ for every closed subspace X of E. We have $\Psi(p) = SpS^{-1}$ for every projection $p \in \mathcal{B}(E)$ and $T \in \mathcal{B}(E) \mapsto STS^{-1}$ is an automorphism of the ring $\mathcal{B}(E)$ which extends Ψ.   □

Question: Do odd automorphisms of the orthoposet of continuous linear projections defined on E extend to anti-automorphisms of the ring $\mathcal{B}(E)$?

A problem in the description of odd automorphisms of lattices of projections is the lack of knowledge about anti-automorphisms of lattices of closed subspaces in the infinite dimensional case. Anti-automorphisms which are ortho-complementations are described, as in the finite dimensional case, by means of symmetric bilinear forms (Varadaradjan, 1985, Lemma 4.2.) but if an anti-automorphism Φ does not satisfy $M \cap \Phi(M) = \{0\}$ then, for a finite dimensional subspace F, $M \mapsto \Phi(M) \cap F$ is not, in general, an anti-automorphism of $[0, F]$ and it is not possible to reduce the infinite dimensional case to the finite dimensional one in the usual way.

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